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Multimagnon scattering in the ferromagnetic XXX–model with inhomogeneities

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Abstract. We determine the transition amplitude for multimagnon scattering induced through an inhomogeneous distribution of the coupling constant in the ferromagnetic XXX-model. The two- and three-particle amplitudes are explicitly calculated at small momenta. This suggests a rather plausible conjecture also for a formula of the general n-particle amplitude.

1. Introduction

We wish to report in this article the calculation of transition amplitudes of multimagnon scattering in the ferromagnetic Heisenberg XXX-chain with an inhomogeneous distribution of the coupling constant.

The Hamiltonian of the model under consideration is given by

$$H = H_{\text{hom}} + H_{\text{inh}} \tag{1}$$

$$H_{\text{hom}} = \frac{J}{4} \sum_{n=-N}^{N} [\boldsymbol{\sigma}_n \boldsymbol{\sigma}_{n+1} - \mathbb{1}] \tag{1a}$$

$$H_{\text{inh}} = \frac{1}{4} \sum_{n=-N}^{N} z_n [\boldsymbol{\sigma}_n \boldsymbol{\sigma}_{n+1} - \mathbb{1}] \tag{1b}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = \sum_{a=1}^{3} \sigma^a \cdot \sigma^a$$

where σ_i^a denotes the Pauli matrices operating in quantum spaces V_i; $i = -N, \ldots, -1, 1, \ldots, N$ attached to a one-dimensional lattice with $2N$ sites, and a periodic boundary condition $V_{N+1} = V_{-N}$ is chosen.

We choose in (1a) the ferromagnetic sign of the coupling constant ($J < 0$) and assume the inhomogeneous piece H_{inh} to be a small perturbation of the homogeneous part H_{hom}, that is, we stipulate for the locally varying couplings z_i

$$|z_i| \ll |J|.$$

The homogeneous XXX-chain is, as a prototype of an integrable model, one of the most thoroughly studied one-dimensional spin models.

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A mathematically rigorous analysis of the model has been provided by Babbitt and Thomas [1]. For a treatment of the XXX-model in the framework of the algebraic Bethe ansatz (ABA) see [2–5]. It is found that the complete spectrum of the model is formed by quasiparticles, here called magnons, and bound states of magnons, the so-called string states. The integrability of the model implies that the interaction between magnons and strings is of a particularly simple structure. It is characterized by the following features [6]:

- multiparticle scattering factorizes into two-particle amplitudes,
- the string states are absolutely stable bound states.

It follows from these properties that neither genuine multiparticle scattering takes place (with a non-trivial reshuffling of the particle momenta) nor does a break-up of the bound states occur.

We will make use of the technique of the ABA [2–5]. A first step in this direction is to embed the Heisenberg spin model (1*a*) into a family of vertex models. The latter models are defined through a monodromy matrix $T(\lambda)$ depending on a spectral parameter λ

$$T(\lambda) = L_N(\lambda) \ldots L_{-N}(\lambda) \tag{2}$$

with the local 'Lax operators' L_n given by

$$L_n(\lambda) = \tfrac{1}{2}[2\mathrm{i}\lambda\, \mathbb{1}_0 \otimes \mathbb{1}_n + \boldsymbol{\sigma}_0 \otimes \boldsymbol{\sigma}_n]. \tag{3}$$

The unit operator $\mathbb{1}_0$ and the Pauli matrices σ_0 act in an auxiliary two-dimensional space, while $\mathbb{1}_n$ and σ_n act in the quantum space V_n. The spin chain model (1*a*) emerges as the logarithmic derivative of the vertex model monodromy matrix

$$H = -J\frac{\mathrm{i}}{2}\left.\frac{d\ln(\mathrm{tr}_0\, T(\lambda))}{d\lambda}\right|_{\lambda=0} - J\frac{2N}{2}\mathbb{1} = \frac{J}{4}\sum_{n=-N}^{N}[\boldsymbol{\sigma}_n\boldsymbol{\sigma}_{n+1} - \mathbb{1}]$$

with tr_0 denoting the trace with respect to the auxiliary space.

The integrability of the vertex models and therewith also the integrability of the XXX-model is based on the fact that there is a *c*-number matrix $R = R(\lambda - \mu)$, such that the Yang–Baxter–Faddeev–Zamolodchikov (YBFZ) relation

$$R(\lambda-\mu)L_i(\lambda) \otimes L_i(\mu) = L_i(\mu) \otimes L_i(\lambda)R(\lambda-\mu) \tag{4}$$

is satisfied. $R = R(\lambda - \mu)$ is in the case at hand given by

$$R(\lambda-\mu) = \begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ 0 & 0 & 0 & f(\mu,\lambda) \end{pmatrix} \tag{5}$$

with

$$f(\mu,\lambda) = 1 + \frac{\mathrm{i}c}{\mu-\lambda} \qquad \text{and} \qquad g(\mu,\lambda) = \frac{\mathrm{i}c}{\mu-\lambda}.$$

The parameter c is set to unity in the following, meaning that the spectral parameter is taken as a dimensionless entity. The local relation (4) induces the global relation

$$R(\lambda-\mu)(T(\lambda)\otimes T(\mu)) = (T(\mu)\otimes T(\lambda))R(\lambda-\mu) \tag{6}$$

which might be considered here as a hallmark of integrability.

The YBFZ relations can be maintained in certain inhomogeneous generalizations of the above models. One possibility is to choose different representations for different sites of the lattice. Such cases have been analysed in [7–9]. A conceptually simpler possibility, noted in [4, 5], consists of attaching local parameters z_i to the local Lax operator, that is, one substitutes $L_i(\lambda)$ by $L_i(\lambda - z_i)$ and obtains then also a modified monodromy matrix

$$T(\lambda, \{z_i\}) = T(\lambda; z_{-N} \ldots, z_N) = \prod_{i=N}^{-N} L_i(\lambda - z_i). \tag{7}$$

One may easily see that the YBFZ relations remain intact,

$$R(\lambda - \mu)\,(T(\lambda, \{z_i\}) \otimes T(\mu, \{z_i\})) = (T(\mu, \{z_i\}) \otimes T(\lambda, \{z_i\}))\,R(\lambda - \mu). \tag{8}$$

It should be noted that equation (8) only holds in general if one specifies for $T(\lambda, \{z_i\})$ and $T(\mu, \{z_i\})$ the same distribution of local parameters $\{z_{-N}, \ldots, z_N\}$. The physics of the model on the other hand appears to be invariant under permutations of the parameters. This is a consequence of the fact that the Bethe ansatz equations, to be mentioned shortly in the following section, which provide the spectrum of the eigenstates, are insensitive to these permutations. It is true (modulo some inessential caveats) that the order of the different representations along the lattice, as mentioned above, is for the same reason irrelevant. We therefore believe that genuine effects of inhomogeneities can only be realized outside the class of integrable models.

It is an easy undertaking to arrive from the inhomogeneous vertex model (7) at the inhomogeneous spin chain (1*b*). Let us make for this purpose the specifications

$$\lambda \to \epsilon\lambda \qquad z_j \to \epsilon z_j. \tag{9}$$

The inhomogeneous Heisenberg magnet (1*b*) is recovered as the logarithmic derivative of the vertex model

$$\begin{aligned} H &= -J\frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\ln(\mathrm{tr}_0\, T(\epsilon\lambda, \{\epsilon z_i\}))|_{\epsilon=0} - \frac{J}{2}\sum_{n=1}^{N}(\lambda - z_n)\,\mathbb{1} \\ &= \frac{J}{4}\sum_{n=1}^{N}(\lambda - z_n)[\boldsymbol{\sigma}_n\boldsymbol{\sigma}_{n+1} - \mathbb{1}] =: \lambda H_0 - H_1. \end{aligned} \tag{10}$$

By taking the derivative with respect to a parameter ϵ, which parametrizes different distributions, one expects to leave the realm of integrability. We will confirm this expectation by evaluating non-vanishing irreducible multiparticle scattering amplitudes which are supposed to vanish identically in integrable models.

We will restrict our considerations to the scattering of elementary magnons (the incorporation of string states is technically definitely much more cumbersome). A simplifying aspect of the problem can be found in that the inhomogeneous perturbation respects the same global $SU(2)$ invariance as the homogeneous term. This implies magnon-number conservation.

The plan of the paper is as follows: in the subsequent section we recall some ingredients of the ABA, discuss the thermodynamic limit and introduce the so-called multisite formalism, taken from [10]. In section 3 we evaluate in first-order perturbation theory multiparticle amplitudes at small momenta. Of crucial importance to achieve this aim will be the representation of form factors as deduced in [10]. The technical tools to be applied in our analysis are approximately the same as those used in [11] for perturbative calculations in antiferromagnetic environments (while of course our calculations are simpler and more simple-minded). The concluding section is devoted to a qualitative discussion of other kinematical regions of multiparticle scattering and a summary. In the appendix we report a perturbative calculation of the spectrum of low-lying states which is conceptually not related to the theme of the bulk of the paper but on a technical level rather similiar to the calculations in section 3.

2. Basics of Bethe ansatz

2.1. Algebraic Bethe ansatz

For the sake of self-consistency we collect here some of the basic aspects of the ABA†.

Let the monodromy matrix (2) be parametrized as

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{11}$$

One deduces from the YBFZ relations (6) the commutators of the operators $A(\lambda), \ldots, D(\lambda)$. Of these 16 relations we only list the following

$$\begin{aligned} &[B(\lambda), B(\mu)] = 0 = [C(\lambda), C(\mu)] \\ &[B(\lambda), C(\mu)] = g(\lambda, \mu)(D(\lambda)A(\mu) - D(\mu)A(\lambda)) \\ &A(\mu)B(\lambda) = f(\mu, \lambda)B(\lambda)A(\mu) + g(\lambda, \mu)B(\mu)A(\lambda) \\ &D(\mu)B(\lambda) = f(\lambda, \mu)B(\lambda)D(\mu) + g(\mu, \lambda)B(\mu)D(\lambda) \\ &C(\lambda)D(\mu) = f(\lambda, \mu)D(\mu)C(\lambda) + g(\mu, \lambda)D(\lambda)C(\mu) \\ &C(\lambda)A(\mu) = f(\mu, \lambda)A(\mu)C(\lambda) + g(\lambda, \mu)A(\lambda)C(\mu). \end{aligned} \tag{12}$$

Let $|0\rangle$ denote the state of highest weight with respect to the tensor product of $SU(2)$ representations in the configuration space $V = \prod_{\otimes} V_i$. This state is annihilated by the operators $C(\lambda)$ and is an eigenstate of the trace of the transfer matrix $\tau(\lambda)$

$$\begin{aligned} &\tau(\lambda) = tr_0 T(\lambda) = A(\lambda) + D(\lambda) \\ &C(\lambda)|0\rangle = 0 \\ &(A(\lambda) + D(\lambda))\,|0\rangle = \left[\left(\mathrm{i}\lambda + \frac{1}{2}\right)^{2N} + \left(\mathrm{i}\lambda - \frac{1}{2}\right)^{2N}\right]|0\rangle. \end{aligned}$$

The ABA renders a representation of the eigenstates of the transfer matrix in terms of the operators $B(\lambda)$—being the Hermitian conjugates of the operators $C(\lambda)$—which act as creation operators of quasiparticles (magnons) on the highest-weight state $|0\rangle$.

Introducing the notation

$$|\Phi(\lambda_1, \ldots, \lambda_l)\rangle = \prod_{n=1}^{l} B(\lambda_n)|0\rangle \tag{13}$$

one arrives—exploiting the commutation relations (12)—at

$$\begin{aligned} (A(\lambda) + D(\lambda)) \prod_{n=1}^{l} B(\lambda_n)|0\rangle &= \Lambda(\lambda; \lambda_1, \ldots, \lambda_l) \prod_{n=1}^{l} B(\lambda_n)|0\rangle \\ &\quad + \sum_{k=1}^{l} \tilde{\Lambda}_k(\lambda_1, \ldots, \lambda_l) B(\lambda) \prod_{j\neq k} B(\lambda_j)|0\rangle \end{aligned} \tag{14}$$

with

$$\Lambda(\lambda; \lambda_1, \ldots, \lambda_l) = \left(\mathrm{i}\lambda + \frac{1}{2}\right)^{2N} \prod_{j=1}^{l} \frac{\lambda_j - \lambda - \mathrm{i}}{\lambda_j - \lambda} + \left(\mathrm{i}\lambda - \frac{1}{2}\right)^{2N} \prod_{j=1}^{l} \frac{\lambda_j - \lambda + \mathrm{i}}{\lambda_j - \lambda} \tag{15}$$

† For a more thorough introduction see [2, 3].

and

$$\tilde{\Lambda}_k(\lambda_1,\ldots,\lambda_l) = g(\lambda_k,\lambda) \times \left[\left(\mathrm{i}\lambda_k + \frac{1}{2}\right)^{2N} \prod_{\substack{j=1\\ j\neq k}}^{l} \frac{\lambda_j - \lambda_k - \mathrm{i}}{\lambda_j - \lambda_k} - \left(\mathrm{i}\lambda_k - \frac{1}{2}\right)^{2N} \prod_{\substack{j=1\\ j\neq k}}^{l} \frac{\lambda_j - \lambda_k + \mathrm{i}}{\lambda_j - \lambda_k} \right]. \tag{16}$$

The second bunch of terms on the right-hand side (r.h.s.) comes from the exchange term in the commutation relations. The spectral parameters have to be specified such that these terms vanish. This gives rise to the Bethe ansatz equations

$$\left(\frac{\mathrm{i}\lambda_k + \frac{1}{2}}{\mathrm{i}\lambda_k - \frac{1}{2}}\right)^{2N} = \prod_{\substack{j=1\\ j\neq k}}^{l} \frac{\lambda_j - \lambda_k + \mathrm{i}}{\lambda_j - \lambda_k - \mathrm{i}}. \tag{17}$$

$|\Phi(\lambda_1,\ldots,\lambda_l)\rangle$ is an eigenstate of the transfer matrix if the last equations are satisfied.

The eigenstates can be classified with respect to the eigenvalue of S^3:

$$S^3|\Phi(\lambda_1,\ldots,\lambda_l)\rangle = \left(\frac{2N}{2} - l\right)|\Phi(\lambda_1,\ldots,\lambda_l)\rangle. \tag{18}$$

The dual wavefunctions are given by

$$\langle\Phi(\{\lambda_j\})| = \langle 0|\prod_{j=1}^{l} B^\dagger(\lambda_j) = (-1)^l \langle 0|\prod_{j=1}^{l} C(\lambda_j). \tag{19}$$

2.2. *Thermodynamical limit*

We introduce suitably normalized operators in order to deal with finite norms of states and finite eigenvalues in the thermodynamical (TD) limit [12]†:

$$\begin{aligned} \tilde{A}(\lambda) &= a^{-1}(\lambda)A(\lambda) \qquad & \tilde{D}(\lambda) &= d^{-1}(\lambda)D(\lambda) \\ \tilde{B}(\lambda) &= a^{-\frac{1}{2}}(\lambda)d^{-\frac{1}{2}}(\lambda)B(\lambda) \qquad & \tilde{C}(\lambda) &= a^{-\frac{1}{2}}(\lambda)d^{-\frac{1}{2}}(\lambda)C(\lambda) \end{aligned} \tag{20}$$

with $a(\lambda) = \alpha(\lambda)^{2N} = (\mathrm{i}\lambda + \frac{1}{2})^{2N}$ the eigenvalue of the operator $A(\lambda)$ and $d(\lambda) = \delta(\lambda)^{2N} = (\mathrm{i}\lambda - \frac{1}{2})^{2N}$ the eigenvalue of $D(\lambda)$ with respect to $|0\rangle$. Thus the operators $\tilde{A}(\lambda)$ and $\tilde{D}(\lambda)$ have correspondingly the eigenvalue 1. Exploiting the relations‡

$$\lim_{N\to\infty} g(\lambda-\mu)\exp\frac{\mathrm{i}(p(\lambda)-p(\mu))N}{2} = -\pi\delta(\lambda-\mu) \tag{21}$$

with $p(\lambda) = \frac{1}{\mathrm{i}}\ln\frac{\mathrm{i}\lambda+\frac{1}{2}}{\mathrm{i}\lambda-\frac{1}{2}}$ denoting the momentum of a magnon and

$$\frac{1}{x} = \frac{1}{x \pm \mathrm{i}\epsilon} \pm \mathrm{i}\pi\delta(x)$$

which hold in the sense of generalized functions [13]—not pointwise—we obtain in the TD limit the simplified relations

$$\begin{aligned} \tilde{A}(\lambda)\tilde{B}(\mu) &= f_-(\lambda,\mu)\tilde{B}(\mu)\tilde{A}(\lambda) \\ \tilde{C}(\lambda)\tilde{D}(\mu) &= f_-(\lambda,\mu)\tilde{D}(\mu)\tilde{C}(\lambda) \end{aligned} \tag{22}$$

with $f_-(\lambda,\mu) = 1 + \frac{\mathrm{i}}{\lambda-\mu-\mathrm{i}\epsilon}$.

† Our prescription differs from the one given in [12] in an inessential way.

‡ Terms of the form $\frac{1}{x}$ have to be evaluated according to the principle-value prescription.

One notes, comparing with equation (12), that the exchange terms have dropped out. The Bethe ansatz equation can therefore be disregarded in the TD limit (as long as one restricts the attention to the sector of elementary magnons). Normalized asymptotic scattering states are generated by acting with creation operators $Z(\lambda) = \tilde{B}(\lambda)\tilde{A}^{-1}(\lambda)$ on the vacuum (the highest-weight state) and are annihilated by operators $Z^{\dagger}(\lambda) = -\tilde{D}^{-1}(\lambda)\tilde{C}(\lambda)$ [6]. The action of the operators $\tilde{A}^{-1}$ and $\tilde{D}^{-1}$ are easily deduced from the relations (22) and from the fact that the vacuum is an eigenstate with unit eigenvalue of $\tilde{A}(\lambda)$ and $\tilde{D}(\lambda)$ and therefore also of $\tilde{A}^{-1}$ and $\tilde{D}^{-1}$.

An incoming scattering state is given by

$$Z(\lambda_1)\ldots Z(\lambda_n)|0\rangle \tag{23}$$

if the rapidities are ordered in such a way that $\lambda_1 < \cdots < \lambda_n$, and represents an outgoing state for $\lambda_1 > \cdots > \lambda_n$.

To relate the incoming to the outgoing states use has to be made of the relation

$$Z(\lambda)Z(\mu) = S(\lambda,\mu)Z(\mu)Z(\lambda) \tag{24}$$

with $S(\lambda,\mu) = \frac{f(\lambda,\mu)}{f(\mu,\lambda)}$ the two-body S-matrix. It is easily seen that the n-magnon S-matrix is given as a product of 2-magnon S-matrices.

The wavefunctions (23) are normalized to delta functions with a unit prefactor.

2.3. *Multisite formalism*

To evaluate scattering amplitudes in the Born approximation, we have to determine form factors [11] of the type

$$\langle 0|\prod_i^l Z^{\dagger}(\lambda_i^C)\mathcal{O}\prod_j^l Z(\lambda_j^B)|0\rangle \tag{25}$$

where the operator $\mathcal{O}$ is given by $\mathcal{O} = \sum_{n=-N}^{N} z_n \boldsymbol{\sigma}_n\boldsymbol{\sigma}_{n+1} \equiv \sum_n \mathcal{O}_{n,n+1}$. So we are led to consider matrix elements of the form

$$\langle 0|\prod_i^l Z^{\dagger}(\lambda_i^C)\mathcal{O}_{n,n+1}\prod_j^l Z(\lambda_j^B)|0\rangle$$
$$= \prod_{i>j} f^{-1}(\lambda_j^C,\lambda_i^C)f^{-1}(\lambda_i^B,\lambda_j^B)\langle 0|\prod_i^l \tilde{C}(\lambda_i^C)\mathcal{O}_{n,n+1}\prod_j^l \tilde{B}(\lambda_j^B)|0\rangle \tag{26}$$

where the latter identity is a straightforward consequence of the definition of Z and the commutation relation (22).

The basic strategy for the determination of the r.h.s. of (26) will consist in decomposing the monodromy matrix into parts as follows:

$$\begin{aligned}
T(\lambda) &= T(3|\lambda)T(2|\lambda)T(1|\lambda)\\
T(1|\lambda) &= L_{n-1}(\lambda)\ldots L_{-N}(\lambda)\\
T(2|\lambda) &= L_{n+1}(\lambda)L_n(\lambda)\\
T(3|\lambda) &= L_N(\lambda)\ldots L_{n+2}(\lambda).
\end{aligned} \tag{27}$$

The submonodromy matrices may be parametrized as $T(\lambda)$ above

$$T(j|\lambda) = \begin{pmatrix} A_j(\lambda) & B_j(\lambda)\\ C_j(\lambda) & D_j(\lambda)\end{pmatrix}$$

with ($j = 1, 2, 3$). The product in (27) is meant to be an ordinary matrix multiplication of 2×2 matrices. The $T(j|\lambda)$ fulfill the global YBFZ commutation relation seperately, acting on the vector space with highest-weight $|0\rangle_j$. The highest-weight state of the total space is given as a tensor product

$$|0\rangle = |0\rangle_3 \otimes |0\rangle_2 \otimes |0\rangle_1. \tag{28}$$

Using the commutation relations, the operators A_j, D_j, which appear if the $B(\lambda)$ are expressed through operators of the subspaces, can be commuted through to the vacuum. This yields the so-called multisite formula [5]

$$\prod_{j=1}^{l^B} B(\lambda_j^B)|0\rangle = \sum_{\{\lambda^{BI}\}\cup\{\lambda^{BII}\}\cup\{\lambda^{BIII}\}=\{\lambda^B\}} \prod_{j_B\in I}^{l_1^B} \prod_{k_B\in II}^{l_2^B} \prod_{l_B\in III}^{l_3^B} K_B \times B_3(\lambda_{l_B}^{BIII})|0\rangle_3 \otimes B_2(\lambda_{k_B}^{BII})|0\rangle_2 \otimes B_1(\lambda_{j_B}^{BI})|0\rangle_1 \tag{29}$$

with

$$K_B = a_2(\lambda_{j_B}^{BI}) d_1(\lambda_{k_B}^{BII}) a_3(\lambda_{j_B}^{BI}) d_1(\lambda_{l_B}^{BIII}) a_3(\lambda_{k_B}^{BII}) d_2(\lambda_{l_B}^{BIII}) f(\lambda_{j_B}^{BI}, \lambda_{k_B}^{BII}) \times f(\lambda_{j_B}^{BI}, \lambda_{l_B}^{BIII}) f(\lambda_{k_B}^{BII}, \lambda_{l_B}^{BIII}).$$

The summation in (29) is with respect to the partition of the set of all Bethe parameters $\{\lambda_j\}$ in three disjunct subsets $\{\lambda^{BI}\}$, $\{\lambda^{BII}\}$ and $\{\lambda^{BIII}\}$ with

$$\mathrm{card}\{\lambda^{BI}\} = l_1^B \qquad \mathrm{card}\{\lambda^{BII}\} = l_2^B \qquad \mathrm{card}\{\lambda^{BIII}\} = l_3^B.$$

A similar representation can be derived for the dual vector $\langle 0| \prod_{j=1}^{l} C(\lambda_j^C)$

$$\langle 0| \prod_{j=1}^{l^C} C(\lambda_j^C) = \sum_{\{\lambda^{CI}\}\cup\{\lambda^{CII}\}\cup\{\lambda^{CIII}\}=\{\lambda^C\}} \prod_{j_C\in I}^{l_1^C} \prod_{k_C\in II}^{l_2^C} \prod_{l_C\in III}^{l_3^C} K_C \times {}_3\langle 0|C_3(\lambda_{l_C}^{CIII}) \otimes \langle 0|_2 C_2(\lambda_{k_C}^{CII}) \otimes \langle 0|_1 C_1(\lambda_{j_C}^{CI}) \tag{30}$$

with

$$K_C = d_2(\lambda_{j_C}^{CI}) a_1(\lambda_{k_C}^{CII}) d_3(\lambda_{j_C}^{CI}) a_1(\lambda_{l_C}^{CIII}) d_3(\lambda_{k_C}^{CII}) a_2(\lambda_{l_C}^{CIII}) \times f(\lambda_{j_C}^{CII}, \lambda_{k_C}^{CI}) f(\lambda_{j_C}^{CIII}, \lambda_{l_C}^{CI}) f(\lambda_{k_C}^{CIII}, \lambda_{l_C}^{CII}).$$

Inserting (29) and (30) into (26) we obtain

$$\langle 0| \prod_{j=1}^{l} C(\lambda_j^C) \mathcal{O}_{n,n+1} \prod_{k=1}^{l} B(\lambda_k^B)|0\rangle = \sum_{I,II,III} \prod_{I,II,III} \prod_{I\leqslant J<K\leqslant III} a_J(\lambda_K^C) \times d_K(\lambda_J^C) a_K(\lambda_J^B) d_J(\lambda_K^B) f(\lambda_J^C, \lambda_K^C) f(\lambda_K^B, \lambda_J^B) \mathcal{S}_{l_1}^{(I)}(\{\lambda_I^C\}, \{\lambda_I^B\}) \times \mathcal{S}_{l_3}^{(III)}(\{\lambda_{III}^C\}, \{\lambda_{III}^B\}) \langle 0|C_2(\lambda_{II}^C) \mathcal{O}_{n,n+1} B(\lambda_{II}^B)|0\rangle \tag{31}$$

with $\mathcal{S}_{l_i}^j(\{\lambda_j^C\}, \{\lambda_k^B\}) = {}_j\langle 0| \prod_i C(\lambda_i^C) \prod_i B\lambda_i^B)|0\rangle_j$ being the scalar product in the jth space. The cardinality of the partition sets $\{\lambda_i^C\}$ and $\{\lambda_i^B\}$ are equal to each other. Matrix elements with $\mathrm{card}\{\lambda_j^C\} \neq \mathrm{card}\{\lambda_k^B\}$ vanish.

Taking into account the normalization of the operators $\tilde{B}$ and $\tilde{C}$ relative to B and C we arrive at the following expression

$$\langle 0| \prod_{j=1}^{l} \tilde{C}(\lambda_j^C) \mathcal{O}_{n,n+1} \prod_{k=1}^{l} \tilde{B}(\lambda_k^B)|0\rangle = \sum_{I,II,III} \prod_{I,II,III} \langle 0|C_2(\lambda_{II}^C) \mathcal{O}_{n,n+1} B_2(\lambda_{II}^B)|0\rangle$$

$$\times \prod_{I\leqslant J<K\leqslant III} f(\lambda_J^C,\lambda_K^C) f(\lambda_K^B,\lambda_J^B)\alpha\delta(\lambda_I^C)^{-\frac{N_1}{2}}\alpha\delta(\lambda_{III}^C)^{-\frac{N_3}{2}}\alpha\delta(\lambda_I^B)^{-\frac{N_1}{2}}$$
$$\times\alpha\delta(\lambda_{III}^B)^{-\frac{N_3}{2}}\alpha\delta(\lambda_{II}^B)^{-1}\alpha\delta(\lambda_{II}^C)^{-1}$$
$$\times\left[\frac{r(\lambda_I^B)}{r(\lambda_I^C)}\right]^{\frac{N_3}{2}+1}\left[\frac{r(\lambda_{III}^C)}{r(\lambda_{III}^B)}\right]^{\frac{N_1}{2}+1}\left[\frac{r(\lambda_{II}^B)}{r(\lambda_{II}^C)}\right]^{\frac{N_3-N_1}{2}}$$
$$\times\mathcal{S}_{l_1}^{(I)}(\{\lambda_I^C\},\{\lambda_I^B\})\mathcal{S}_{l_3}^{(III)}(\{\lambda_{III}^C\},\{\lambda_{III}^B\}) \tag{32}$$

where $\alpha\delta(\lambda)\equiv\alpha(\lambda)\delta(\lambda)$ and $r(\lambda)\equiv\frac{\alpha(\lambda)}{\delta(\lambda)}$.

One has the following recursion relation for scalar products [10]:

$$\mathcal{S}_l(\{\lambda_j^C\},\{\lambda_k^B\}) = a(\lambda_1^C)\sum_{n=1}^{l} d(\lambda_n^B)g(\lambda_1^C,\lambda_n^B)\prod_{j\neq1}^{l} g(\lambda_1^C,\lambda_j^C)\prod_{k\neq n}^{l} g(\lambda_k^B,\lambda_n^B)\mathcal{S}_{l-1}(\hat a_1(\lambda),\hat d_1(\lambda))$$
$$+d(\lambda_1^C)\sum_{n=1}^{l} a(\lambda_n^B)g(\lambda_n^B,\lambda_1^C)\prod_{j\neq1}^{l} g(\lambda_j^C,\lambda_j^C\lambda_1^C)$$
$$\times\prod_{k\neq n}^{l} g(\lambda_n^B,\lambda_k^B)\mathcal{S}_{l-1}(\hat a_2(\lambda),\hat d_2(\lambda))$$

with $\hat a_1(\lambda) = a(\lambda)h(\lambda,\lambda_n^B), \hat a_2(\lambda) = a(\lambda)h(\lambda,\lambda_1^C)$ and $\hat d_1(\lambda) = d(\lambda)h(\lambda_1^C,\lambda)$, $\hat d_2(\lambda) = d(\lambda)h(\lambda_n^B,\lambda)$, while $h(\lambda,\mu) = \frac{g(\lambda,\mu)}{f(\lambda,\mu)} = 1+\frac{\lambda-\mu}{\mathrm{i}}$. We have quoted here on the r.h.s. the functional dependence of the scalar products on the vacuum eigenvalues which have changed going from the left-hand side (l.h.s.) to the r.h.s. from $a(\lambda)$ to $\hat a(\lambda)$ and $d(\lambda)$ to $\hat d(\lambda)$ respectively, which makes the solution of the recursion relation difficult in general. The two-term recursion relation simplifies in the TD limit, if we concentrate on the irreducible part of the amplitude.

We obtain for the normalized scalar product in the limit $N\to\infty$:

$$\lim_{N\to\infty}\prod_{CI}^{l_1}\prod_{BI}^{l_1}\left[\frac{r(\lambda_I^B)}{r(\lambda_I^C)}\right]^{\frac{N_3}{2}+1}\alpha\delta(\lambda_I^C)^{-\frac{N_1}{2}}\alpha\delta(\lambda_I^B)^{-\frac{N_1}{2}}\mathcal{S}_{l_1}^{(I)}(\{\lambda_I^C\},\{\lambda_I^B\})$$
$$=\left\{\sum_{n_1=1}^{l_1}\left[\frac{r(\lambda_{n_1}^{BI})}{r(\lambda_{n_1}^{CI})}\right]^{\frac{N_3-N_1}{2}+1} g(\lambda_1^{CI},\lambda_{n_1}^{BI})\prod_{j_1\neq1}^{l_1} g(\lambda_1^{CI},\lambda_{j_1}^{CI})\prod_{k_1\neq n_1}^{l_1} g(\lambda_{k_1}^{BI},\lambda_{n_1}^{BI})\right.$$
$$\left.-\pi\sum_{n_1=1}^{l_1}\delta(\lambda_1^{CI}-\lambda_{n_1}^{BI})\prod_{j_1\neq1}^{l_1} g(\lambda_{j_1}^{CI},\lambda_1^{CI})\prod_{k_1\neq n_1}^{l_1} g(\lambda_{n_1}^{BI},\lambda_{k_1}^{BI})\right\}$$
$$\times\lim_{N\to\infty}\prod_{CI\neq1}^{l_1}\prod_{BI\neq n_1}^{l_1}\left[\frac{r(\lambda_I^B)}{r(\lambda_I^C)}\right]^{\frac{N_3}{2}+1}\alpha\delta(\lambda_I^C)^{-\frac{N_1}{2}}\alpha\delta(\lambda_I^B)^{-\frac{N_1}{2}}$$
$$\times\mathcal{S}_{l_1}^{(I)}(a(\lambda)h(\lambda,\lambda_{n_1}^{BI}),d(\lambda)h(\lambda_1^{CI})) \tag{33}$$

where we have used relation (21). One should note that the second term on the r.h.s. only contributes—due to the appearance of a delta function—to scattering processes where at least one magnon goes over unscattered from the incoming to the outgoing state. The irreducible scattering amplitude, however, refers by definition to that part of the amplitude from which all energy conserving subprocesses have been subtracted. It means that the restriction to the irreducible amplitudes effectively implies that only the first term of the recursion relation (33) has to be taken into account for the evaluation of $\mathcal{S}^{(I)}$. The ensuing one-term recursion relation is easily solved with the result

$$\lim_{N\to\infty}\prod_{CI}^{l_1}\prod_{BI}^{l_1}\left[\frac{r(\lambda_I^B)}{r(\lambda_I^C)}\right]^{\frac{N_3}{2}+1}\alpha\delta(\lambda_I^C)^{-\frac{N_1}{2}}\alpha\delta(\lambda_I^B)^{-\frac{N_1}{2}}\mathcal{S}_{l_1}^{(I)}(\{\lambda_I^C\},\{\lambda_I^B\})_{\rm irr}$$
$$=\sum_{n_1=1}^{l_1}\sum_{\substack{n_2=1\\ n_2\neq n_1}}^{l_1}\cdots\sum_{\substack{n_{l_1}=1\\ n_{l_1}\neq n_1,\dots,n_{l_1-1}}}\prod_{j>i=1}^{l_1}g(\lambda_i^{CI},\lambda_j^{CI})$$
$$\times\prod_{k_1\neq n_1}^{l_1}g(\lambda_{k_1}^{BI},\lambda_{n_1}^{BI})\ldots\prod_{k_{l_1}\neq n_1,\dots,n_{l_1}}^{l_1}g(\lambda_{k_{l_1}}^{BI},\lambda_{n_{l_1}}^{BI})$$
$$\times\prod_{k>i=1}^{l_1}[h(\lambda_k^{CI},\lambda_{n_i}^{BI})h(\lambda_{n_k}^{BI},\lambda_i^{CI})]\prod_{k=1}^{l_1}\left[\frac{r(\lambda_I^B)}{r(\lambda_I^C)}\right]^{\frac{N_3}{2}+1}g(\lambda_k^{CI},\lambda_{n_k}^{BI}) \tag{34}$$

with the subscript 'irr' indicating the restriction to that part of the scalar product that contributes finally to the irreducible amplitude. The extension of the r.h.s. of equation (34) by factors $1=\frac{g(\lambda,\mu)}{g(\lambda,\mu)}$ enables us to represent it as a determinant multiplied by some overall factor:

$$\lim_{N\to\infty}\prod_{CI}^{l_1}\prod_{BI}^{l_1}\left[\frac{r(\lambda_I^B)}{r(\lambda_I^C)}\right]^{\frac{N_3}{2}+1}\alpha\delta(\lambda_I^C)^{-\frac{N_1}{2}}\alpha\delta(\lambda_I^B)^{-\frac{N_1}{2}}\mathcal{S}_{l_1}^{(I)}(\{\lambda_I^C\},\{\lambda_I^B\})_{\rm irr}$$
$$=\prod_{j>i}^{l_1}g(\lambda_i^{CI},\lambda_j^{CI})\prod_{j>i}^{l_1}g(\lambda_j^{BI},\lambda_i^{BI})\prod_{i,j}^{l_1}h(\lambda_i^{CI},\lambda_j^{BI})\prod_{i}^{l_1}\left[\frac{r(\lambda_i^{BI})}{r(\lambda_i^{CI})}\right]^{\frac{N_3}{2}+1}\det\nolimits_{l_1}\mathcal{M}^{(I)} \tag{35}$$

with $\mathcal{M}_{ij}^{(I)}=\frac{g(\lambda_i^{CI},\lambda_j^{BI})}{h(\lambda_i^{CI},\lambda_j^{BI})}$.

A similar relation holds for the part denoted with III (here only the second term in the recursion relation of the scalar product contributes):

$$\lim_{N\to\infty}\prod_{CIII}^{l_3}\prod_{BIII}^{l_3}\left[\frac{r(\lambda_{III}^C)}{r(\lambda_{III}^B)}\right]^{\frac{N_1}{2}+1}\alpha\delta(\lambda_{III}^C)^{-\frac{N_3}{2}}\alpha\delta(\lambda_{III}^B)^{-\frac{N_3}{2}}\mathcal{S}_{l_3}^{(III)}(\{\lambda_{III}^C\},\{\lambda_{III}^B\})_{\rm irr}$$
$$=\prod_{j>i}^{l_3}g(\lambda_j^{CIII},\lambda_i^{CIII})g(\lambda_i^{BIII},\lambda_j^{BIII})\prod_{i,j}^{l_3}h(\lambda_i^{BIII},\lambda_j^{CIII})$$
$$\times\prod_{i}^{l_3}\left[\frac{r(\lambda_i^{CIII})}{r(\lambda_i^{BIII})}\right]^{\frac{N_1}{2}+1}\det\nolimits_{l_3}\mathcal{M}^{(III)} \tag{36}$$

with $\mathcal{M}_{ij}^{(III)}=\frac{g(\lambda_j^{BIII},\lambda_i^{CIII})}{h(\lambda_j^{BIII},\lambda_i^{CIII})}$.

Inserting (35) and (36) into (31) we obtain

$$\langle 0|\prod_{j=1}^{l}\tilde{C}(\lambda_j^C)\mathcal{O}_{n,n+1}\prod_{k=1}^{l}\tilde{B}(\lambda_k^B)|0\rangle_{\rm irr}=\prod_{i}^{l}\left[\frac{r(\lambda_i^C)}{r(\lambda_i^B)}\right]^{\frac{N_1-N_3}{2}}\prod_{j>i}^{l}g(\lambda_j^C,\lambda_i^C)g(\lambda_i^B,\lambda_j^B)$$
$$\times\sum_{I,II,III}(-1)^{[P_C]+[P_B]}\langle 0|\prod_{II}\tilde{C}_2(\lambda_{II}^C)\mathcal{O}_{n,n+1}\prod_{II}\tilde{B}_2(\lambda_{II}^B)|0\rangle$$

$$\times \prod_{I\leqslant J<K\leqslant III} h(\lambda_K^C,\lambda_J^C)h(\lambda_J^B,\lambda_K^B)\prod_{i,j}^{l_1} h(\lambda_i^{CI},\lambda_j^{BI})\prod_{i,j}^{l_3} h(\lambda_i^{BIII},\lambda_j^{CIII})$$
$$\times \det(R^{-1}(\lambda_I^C)\mathcal{M}^{(I)}(\lambda_I^C,\lambda_I^B)R(\lambda_I^B))$$
$$\times \det(R(\lambda_{III}^C)\mathcal{M}^{(III)}(\lambda_{III}^C,\lambda_{III}^B)R^{-1}(\lambda_{III}^B)) \quad (37)$$

with $R(\lambda)_{ij} = r(\lambda_i)\delta_{ij}$. While deriving this result we used $f(\lambda,\mu) = g(\lambda,\mu)h(\lambda,\mu)$ and the antisymmetry of the g's.

$[P_B]$ stands for the parity of the permutation

$$P_B : \{\lambda_{II}^B\}\cup\{\lambda_I^B\}\cup\{\lambda_{III}^B\} \to \{\lambda^B\}$$

while $[P_C]$ stands for the parity of the permutation

$$P_C : \{\lambda_{II}^C\}\cup\{\lambda_I^C\}\cup\{\lambda_{III}^C\} \to \{\lambda^C\} \quad (38)$$

with the enumeration in each subset according to the original one.

It is possible, in principle, to write the result in a more compact way, namely as the determinant of the sum of three matrices [14]. As it is not useful for our purpose we will not pursue this line of reasoning.

3. Low-energy limit

To start with let us make the simplifications which are due to the special form of the perturbation. The matrix element

$${}_2\langle 0|\prod_i \tilde{C}_2(\lambda_i^C)\boldsymbol{\sigma}_n\boldsymbol{\sigma}_{n+1}\prod_i \tilde{B}_2(\lambda_i^B)|0\rangle_2 \quad (39)$$

appearing in equation (37) is to be evaluated with respect to the two-site highest-weight state $|0\rangle_2$. Therefore, at most two operators B and C can show up in (39) (applying two operators B_2 to $|0\rangle_2$ one reaches the state of lowest weight of the two-site vector space). Since we are restricting our attention to scattering events in non-forward directions we may evaluate instead of (39) the matrix element

$${}_2\langle 0|\prod_i \tilde{C}_2(\lambda_i^C)\left(\boldsymbol{\sigma}_n\boldsymbol{\sigma}_{n+1} - \mathbb{1}_n\cdot\mathbb{1}_{n+1}\right)\prod_i \tilde{B}_2(\lambda_i^B)|0\rangle_2 \quad (40)$$

with $\mathbb{1}_n$ the identity in V_n (the addition of $\mathbb{1}_n\cdot\mathbb{1}_{n+1}$ gives only a contribution to the amplitude in the forward direction). However, (40) vanishes on the state of highest weight (no operators B_2 and C_2) and on the state of lowest weight (two operators B_2 and C_2). We are left with the matrix element with one operator B_2 and C_2, which is straightforwardly calculated

$${}_2\langle 0|\tilde{C}_2(\lambda_{II}^C)(\boldsymbol{\sigma}_n\boldsymbol{\sigma}_{n+1} - \mathbb{1}_n\cdot\mathbb{1}_{n+1})\tilde{B}_2(\lambda_{II}^B)|0\rangle_2 = 2\frac{1}{\alpha\delta(\lambda_{II}^B)}\frac{1}{\alpha\delta(\lambda_{II}^C)}. \quad (41)$$

Taking the normalization and the last result into account we obtain for the transition amplitude the representation

$$\sum_n \langle 0|\prod_{j=1}^{l} Z^\dagger(\lambda_j^C)\mathcal{O}_{n,n+1}\prod_{k=1}^{l} Z(\lambda_k^B)|0\rangle = \sum_n z_n 2\prod_i^l\left[\frac{r(\lambda_i^C)}{r(\lambda_i^B)}\right]^{\frac{N_1-N_3}{2}}\prod_{j>i}^{l}\frac{g(\lambda_j^C,\lambda_i^C)}{f(\lambda_j^C,\lambda_i^C)}\frac{g(\lambda_i^B,\lambda_j^B)}{f(\lambda_i^B,\lambda_j^B)}$$
$$\times \sum_{I,II,III}{}' (-1)^{[P_C]+[P_B]}\frac{1}{\alpha\delta(\lambda_{II}^B)}\frac{1}{\alpha\delta(\lambda_{II}^C)}\prod_{I\leqslant J<K\leqslant III} h(\lambda_K^C,\lambda_J^C)h(\lambda_J^B,\lambda_K^B)$$

$$\times \prod_{i,j}^{l_1} h(\lambda_i^{CI}, \lambda_j^{BI}) \prod_{i,j}^{l_3} h(\lambda_i^{BIII}, \lambda_j^{CIII}) \det(R^{-1}(\lambda_I^C)\mathcal{M}^{(I)}(\lambda_I^C, \lambda_I^B)R(\lambda_I^B))$$
$$\times \det(R(\lambda_{III}^C)\mathcal{M}^{(III)}(\lambda_{III}^C, \lambda_{III}^B)R^{-1}(\lambda_{III}^B)). \tag{42}$$

The slash on the sum over the partitions is supposed to indicate that only partitions with exactly one representative present in the subset labelled by II are to be taken.

We are now prepared to examine the behaviour of irreducible scattering amplitudes at low momenta ($\lambda_i \sim p_i$ for small momentum) with two or more magnons involved (the one-particle amplitude will be quoted below for the sake of completeness). An obvious method to get a handle on formula (42) consists in a systematic expansion in powers of momenta, as far as they appear in functions h and keeping at the same time the functions g unexpanded. The leading term is obtained by putting h consistently to one at all places where it appears in (42). This yields for $\lambda_\alpha \in \{\lambda^C\}$

$$\sum_n \langle 0| \prod_{j=1}^{l} Z^\dagger(\lambda_j^C)\mathcal{O}_{n,n+1} \prod_{k=1}^{l} Z(\lambda_k^B)|0\rangle \approx \sum_n 32 z_n \prod_{j>i}^{l} \frac{g(\lambda_j^C, \lambda_i^C)}{f(\lambda_j^C, \lambda_i^C)} \frac{g(\lambda_i^B, \lambda_j^B)}{f(\lambda_i^B, \lambda_j^B)}$$
$$\times \sum_{I,II,III}{}' (-1)^{[P_C]+[P_B]} \det g(\lambda_I^C, \lambda_I^B) \det g(\lambda_{III}^B, \lambda_{III}^C). \tag{43}$$

The prefactors $\prod_{j>i}^{l} \frac{g(\lambda_i,\mu_j)}{f(\lambda_i,\mu_j)}$ may also be put equal to one in leading order by noting that $\frac{g(\lambda,\mu)}{f(\lambda,\mu)} \approx 1 + \mathrm{O}(\lambda - \mu)$. The sum over the partitions I and III in (43) renders a vanishing result as one infers from the Laplace formula for the determinant of a sum of matrices [5]:

$$\det(A + B) = \sum_{P_L, P_C} (-1)^{[P_L]+[P_C]} \det A_{P_L,P_C} \det B_{P_L,P_C} \tag{44}$$

where P_L is the partition of rows in subsets of rows of A and B, while P_C is analogous the partition of columns, and the fact that g is odd

$$g(\lambda_i^C, \lambda_j^B) + g(\lambda_j^B, \lambda_i^C) = 0.$$

For the next order of the expansion in powers of momenta we obtain in a straightforward manner the following result:

$$\langle 0| \prod_{j=1}^{l} Z^\dagger(\lambda_j^C) z_i \boldsymbol{\sigma}_i \boldsymbol{\sigma}_{i+1} \prod_{k=1}^{l} Z(\lambda_k^B)|0\rangle \approx 2i \prod_{j>i}^{l} h^{-1}(\lambda_j^C, \lambda_i^C) h^{-1}(\lambda_i^B, \lambda_j^B) \prod_i \left[\frac{r(\lambda_i^C)}{r(\lambda_i^B)}\right]^{\frac{N_1-N_3}{2}}$$
$$\times 16 z_i \sum_C \lambda_\alpha \sum_{I,II,III}{}' (-1)^{[P_B]+[P_C]} (l_3 - l_1 + 6\epsilon_\alpha^{I,III})$$
$$\times \det g(\lambda_I^C, \lambda_I^B) \det g(\lambda_{III}^B, \lambda_{III}^C) \tag{45}$$

with

$$\epsilon_\alpha^{I,III} = \begin{cases} +1; \lambda_\alpha \in I \\ -1; \lambda_\alpha \in III \\ 0; \lambda_\alpha \in II. \end{cases}$$

If $\lambda_\alpha \in \{\lambda^B\}$ we obtain the same result up to an overall minus sign.

The result can be simplified further, using the following chain of identities:

$$\sum_{I,III} (-1)^{[P_B]+[P_C]} (l_3 - l_1) \det g(\lambda_I^C, \lambda_I^B) \det g(\lambda_{III}^B, \lambda_{III}^C)$$

$$= \frac{\partial}{\partial x} \sum_{I,III} (-1)^{[P_B]+[P_C]} \det g(x\lambda_I^C, x\lambda_I^B) \det g(x^{-1}\lambda_{III}^B, x^{-1}\lambda_{III}^C)|_{x=1}$$
$$= \frac{\partial}{\partial x} \det(g(x\lambda^C, x\lambda^B) + g(x^{-1}\lambda^B, x^{-1}\lambda^C))|_{x=1}$$
$$= \left. \frac{\partial}{\partial x} \left(\frac{1}{x} - x\right)^{l_1+l_3} \right|_{x=1} \det g(\lambda^C, \lambda^B)$$
$$= 0 \qquad \text{for } l_1 + l_3 > 1$$

where we used again the Laplace formula and the antisymmetry of the g's.

The remaining term of the first-order Taylor expansion is

$$\langle 0| \prod_{j=1}^{l} Z^\dagger(\lambda_j^C) z_i \boldsymbol{\sigma}_i \boldsymbol{\sigma}_{i+1} \prod_{k=1}^{l} Z(\lambda_k^B)|0\rangle = \prod_{j>i}^{l} h^{-1}(\lambda_j^C, \lambda_i^C) h^{-1}(\lambda_i^B, \lambda_j^B) \prod_i \left[\frac{r(\lambda_i^C)}{r(\lambda_i^B)}\right]^{\frac{N_1-N_3}{2}}$$
$$\times 16 \cdot 6i z_i \sum_C \lambda_\alpha \sum_{I,II,III}' (-1)^{[P_B]+[P_C]} (\epsilon_\alpha^{I,III}) \det g(\lambda_I^C, \lambda_I^B) \det g(\lambda_{III}^B, \lambda_{III}^C).$$

The sum over the first and the third partition can be combined to a determinant of the sum of two matrices

$$\sum_{I,,III} (-1)^{[P_B]+[P_C]} (\epsilon_\alpha^{I,III}) \det g(\lambda_I^C, \lambda_I^B) \det g(\lambda_{III}^B, \lambda_{III}^C) = \det(g(\lambda^C, \lambda^B) + \hat{g}_\alpha(\lambda^B, \lambda^C)).$$

The matrix $\hat{g}_\alpha(\lambda^B, \lambda^C)$ differs from $g(\lambda^B, \lambda^C)$ in that the αth row is multiplied by (-1). The sum of the two matrices is thus a matrix with only one row of non-vanishing entries. The determinant is again zero except for $l_1 + l_3 = 1$. We keep as a net result that the first-order term of the Taylor expansion gives a non-vanishing contribution only for the one- and two-particle amplitude.

We turn now to the second-order contribution. The computation is tedious, but it proceeds otherwise along the same lines as the first-order calculation. We thus only quote the result (omitting the prefactors):

$$\langle 0| \prod_{j=1}^{l} Z^\dagger(\lambda_j^C) z_i \boldsymbol{\sigma}_i \boldsymbol{\sigma}_{i+1} \prod_{k=1}^{l} Z(\lambda_k^B)|0\rangle \sim 16\tfrac{1}{2} z_i \sum_{\alpha\beta} \lambda_\alpha \lambda_\beta \sum_{I,II,III}' (-1)^{[P_B]+[P_C]}$$
$$\times [\mathrm{i}\epsilon_\alpha^{B,C}(l_1 - l_3 - 6\epsilon_\alpha^{I,III}) \mathrm{i}\epsilon_\beta^{B,C}(l_1 - l_3 - 6\epsilon_\beta^{I,III})$$
$$+ \delta_{\alpha\beta}(34\tilde{\epsilon}^{I,III} + l_1 + l_3)] \det g(\lambda_I^C, \lambda_I^B) \det g(\lambda_{III}^B, \lambda_{III}^C) \qquad (46)$$

with

$$\epsilon_\alpha^{B,C} = \begin{cases} +1 & \lambda_\alpha \in B \\ -1 & \lambda_\alpha \in C \end{cases}$$

and

$$\tilde{\epsilon}^{I,III} = \begin{cases} +1 & \lambda \text{ in } I, III \\ 0 & \text{otherwise.} \end{cases}$$

When $l_1 + l_3 > 2$ this term vanishes as can be shown by generalizing the considerations used in first-order calculation:

• terms proportional to $(l_1 + l_3 + \text{constant})$ vanish by the same argument as used in the first-order calculation (even for $l_1 + l_3 = 2$),

• terms proportional to $(l_1 - l_3)^2$ vanish when regarded as the second derivative with respect to x at $x = 1$ (this term renders for $l_1 + l_3 = 2$ the only non-vanishing contribution),

• terms proportional to $\epsilon_\alpha^{I,III}(l_1 - l_3)$ give a matrix with a prefactor $(\frac{1}{x} - x)^{l_1+l_3-1}(\frac{1}{x} + x)$ of which the derivative with respect to x at $x = 1$ vanishes,

• terms proportional to $\epsilon_\alpha^{I,III}\epsilon_\beta^{I,III}$ give—after applying the Laplace formula—a matrix with at most two columns or rows or one column and one row not zero.

We finish this section by quoting the leading terms of the transition amplitudes at small momenta with the explicit expressions for $l \leqslant 3$ in lowest order:

• $l = 1$

$$\sum_n z_n \langle 0|Z^\dagger(\lambda)\boldsymbol{\sigma}_n\boldsymbol{\sigma}_{n+1}Z(\mu)|0\rangle = -2\sum_n z_n \left[\frac{r(\lambda)}{r(\mu)}\right]^{\frac{N_1-N_3}{2}} \frac{1}{\lambda^2 + \frac{1}{4}}\frac{1}{\mu^2 + \frac{1}{4}} \tag{47}$$

• $l = 2$, cf equation (45)

$$\sum_n z_n \langle 0|Z^\dagger(\lambda_1^C)Z^\dagger(\lambda_2^C)\boldsymbol{\sigma}_n\boldsymbol{\sigma}_{n+1}Z(\lambda_1^B)Z(\lambda_2^B)|0\rangle$$
$$\approx -64\hat{\mathcal{Z}}\left(\sum_C \lambda^C - \sum_B \lambda^B\right)\left(\sum_C \lambda^C - \sum_B \lambda^B\right)^2 \det{}_2\left(\frac{1}{\lambda^C - \lambda^B}\right) \tag{48}$$

• $l = 3$, cf equation (46)

$$\sum_n z_n \langle 0|\prod_{i=1}^{3} Z^\dagger(\lambda_i^C)\boldsymbol{\sigma}_n\boldsymbol{\sigma}_{n+1}\prod_{i=1}^{3} Z(\lambda_i^B)|0\rangle \approx -128\hat{\mathcal{Z}}\left(\sum_C \lambda^C - \sum_B \lambda^B\right)$$
$$\times\left(\sum_C \lambda^C - \sum_B \lambda^B\right)^3 \det{}_3\left(\frac{1}{\lambda^C - \lambda^B}\right) \tag{49}$$

with $\det_l(\frac{1}{\lambda^C - \lambda^B})$ denoting the Cauchy determinant of a $l \times l$ matrix

$$\det{}_l\left(\frac{1}{\lambda_i - \mu_j}\right) = (-1)^{\frac{l(l-1)}{2}} \frac{\prod_{i<j}^{l}(\lambda_i - \lambda_j)\prod_{i<j}^{l}(\mu_i - \mu_j)}{\prod_{i,j}^{l}(\lambda_i - \mu_j)}$$

and

$$\hat{\mathcal{Z}}\left(\sum_C \lambda^C - \sum_B \lambda^B\right) = \sum_n z_n \prod_{i=1}^{l}\left[\frac{r(\lambda_i^C)}{r(\lambda_i^B)}\right]^{\frac{N_1-N_3}{2}}$$

the Fourier transform of the distribution of couplings.

Some remarks may be in order.

(1) The one-particle amplitude quoted above is in fact the full Born term (not the leading piece at small momentum).

(2) The Fourier transform $\hat{\mathcal{Z}}$ of the distribution of the coupling constant appears in equations (48) and (49), which is a function of the difference of the ingoing and outgoing momenta. A homogeneous addition to the distribution $\{z_n\}$ should not and will not have an effect on the formulae since such an addition will render a contribution proportional to $\delta(\sum_C \lambda^C - \sum_B \lambda^B)$ which is annihilated by the powers of momenta appearing in (48) and (49).

(3) To apply the above expressions to physical processes of magnon scattering one has to restrict the respective expressions to the energy shell, given by $\sum_i {\lambda_i^B}^2 = \sum_i {\lambda_i^C}^2$.

4. Conclusion

The main result of this paper are the formulae (48) and (49) for two- and three-magnon scattering at small momenta. An obvious generalization to l-particle scattering may be conjectured:

$$\langle 0|\prod_{i=1}^{l} Z^{\dagger}(\lambda_i^C)\boldsymbol{\sigma}_i\boldsymbol{\sigma}_{i+1}\prod_{i=1}^{l} Z(\lambda_i^B)|0\rangle \approx -16.2^l \hat{\mathcal{Z}}\left(\sum_C \lambda^C - \sum_B \lambda^B\right) \times\left(\sum_C \lambda^C - \sum_B \lambda^B\right)^l \det_l\left(\frac{1}{\lambda^C-\lambda^B}\right). \tag{50}$$

We are not able to prove this conjecture so far.

It may be noted that the nominator of the determinant $\prod_{i<j}^{l}(\lambda_i^B-\lambda_j^B)\prod_{i<j}^{l}(\lambda_i^C-\lambda_j^C)$ appearing on the r.h.s. of equation (50) reflects the Pauli exclusion principle realized by the Bethe wave states [15]. While the overall zero degree of homogeneity with respect to uniform scaling of all momenta may be plausible, we are not aware of an *a priori* explanation for the appearance of the lth power of the difference of momenta of the incoming and outgoing particles. We speculate that this reflects genuinely the integrability of the homogeneous XXX-model.

There are other kinematical regions besides the one of low momenta for which simple and reliable estimates can be made. If all momenta and all differences of momenta become large, the n-particle transition amplitude decreases with $\rho^{-(n+3)}$—ρ denoting a common scale of all momenta—as can be inferred from an inspection of equation (37). An interesting kinematical region—also accessible to a rather detailed analytical description—is given by the setting

$$|\lambda_i^B-\lambda_j^B|\ll 1 \qquad |\lambda_i^C-\lambda_j^C|\ll 1 \qquad |\lambda_i^B-\lambda_j^C|\gg 1 \qquad \forall i,j$$
$$|\lambda^B|\sim|\lambda^C|\sim\rho\gg 1.$$

This situation is realized if a bunch of particles travelling approximately with the same velocity is collectively scattered backwards at the inhomogeneity.

The piece of (37) supplying the ρ dependence in this case is given by

$$_2\langle 0|\tilde{C}_2(\lambda_{II}^C)\mathcal{O}_{n,n+1}\tilde{B}_2(\lambda_{II}^B)|0\rangle_2\prod_{i,j}^{l_1} h(\lambda_i^{CI},\lambda_j^{BI})\prod_{i,j}^{l_3} h(\lambda_i^{BIII},\lambda_j^{CIII}) \times\det(\mathcal{M}^{(I)}(\lambda_I^C,\lambda_I^B))\det(\mathcal{M}^{(III)}(\lambda_{III}^C,\lambda_{III}^B))$$

for which one easily calculates the scaling behaviour $\rho^{-(n+3)}$.

To arrive at this conclusion it is essential to view the determinants in the above formula as derivatives of Cauchy determinants:

$$\det\mathcal{M}_{ij}\approx\det_l\frac{1}{(\lambda_i^C-\lambda_j^B)^2}=\frac{\partial}{\partial\lambda_1^C}\cdots\frac{\partial}{\partial\lambda_l^C}\det_l\frac{1}{(\lambda_i^C-\lambda_j^B)}.$$

A completely open problem within our approach is the treatment of string states. The determination of break-up amplitudes for string states seems to us a particularly challenging problem.

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Appendix

In this appendix we use the result (31) to determine the shift of energy eigenvalues caused by $H_{\rm inh} = \frac{1}{4}\sum_{n=1}^{N} z_n[\boldsymbol{\sigma}_n\boldsymbol{\sigma}_{n+1} - \mathbb{1}]$ [16, 17].

The lowest excitation is generated by flipping one spin ($l = 1$). The solution of the Bethe ansatz equation is in this case

$$\lambda = \frac{1}{2}\cot\frac{p_0}{2} \qquad p_0 = \frac{2\pi k}{N} \qquad k = 1, \ldots, N. \tag{51}$$

Taking into account parity degeneracy the first-order correction to the energy $E^{(0)}(\lambda) = \frac{1}{2}\frac{1}{\lambda^2+\frac{1}{4}}$ is found to be

$$E^{(1)} = \frac{\mathcal{V}(\lambda,\lambda)}{\langle 0|C(\lambda)B(\lambda)|0\rangle} \pm 2\frac{|\mathcal{V}(\lambda,-\lambda)|}{\langle 0|C(\lambda)B(\lambda)|0\rangle}$$

with $\mathcal{V}(\mu_1,\mu_2) = -\frac{1}{4}\sum_{j=1}^{N} z_j\langle 0|C(\mu_1)(\boldsymbol{\sigma}_j\boldsymbol{\sigma}_{j+1} - \mathbb{1})B(\mu_2)|0\rangle$ which leads to

$$E^{(1)}(\lambda) = E^{(0)}(\lambda)\left[\left(\frac{1}{N}\sum_{j=1}^{N} z_j\right) \pm \sqrt{\sum_{j,k=1}^{N}\frac{z_j z_k}{N^2}\exp(-2\mathrm{i}p_0(\lambda)(j-k))}\right]. \tag{52}$$

This shows that the energy correction depends in first-order both on the mean-value of the couplings $\overline{z} = \frac{1}{N}\sum_{j=1}^{N} z_j$ and on the Fourier transform of the distribution $\frac{1}{N}\sum_{j=1}^{N} z_j\exp(\pm 2\mathrm{i}p_0 j)$ (here $p_0(\lambda) = \frac{1}{\mathrm{i}}\ln\frac{\mathrm{i}\lambda+\frac{\mathrm{i}}{2}}{\mathrm{i}\lambda-\frac{\mathrm{i}}{2}}$).

The second-order corrections can be obtained from the secular equation

$$E_n^{(2)}(\lambda) = \sum_{m\neq n}\frac{\tilde{\mathcal{V}}_{nm}\tilde{\mathcal{V}}_{mn}}{E_n^{(0)} - E_m^{(0)}}$$

where the matrix elements are taken with respect to the corrected wavefunction in the zeroth approximation

$$\begin{aligned}\tilde{\mathcal{V}}_{nm} &= \langle c_1^{(0)\star}\Phi(\lambda) \pm c_2^{(0)\star}\Phi(-\lambda)|H_1|\Phi(\mu)\rangle \\ &= c_1^{(0)\star}\mathcal{V}(\lambda,\mu) \pm c_2^{(0)\star}\mathcal{V}(-\lambda,\mu)\end{aligned}$$

with $\mathcal{V}(\lambda,\mu)$ defined as in (4) and $c^{(0)}$ being the following expressions

$$\begin{aligned}c_1^{(0)} &= \sqrt{\frac{\mathcal{V}(\lambda,-\lambda)}{2|\mathcal{V}(\lambda,-\lambda)|}} \\ c_2^{(0)} &= \sqrt{\frac{\mathcal{V}(-\lambda,\lambda)}{2|\mathcal{V}(-\lambda,\lambda)|}}.\end{aligned}$$

Inserting the explicit formulae yields

$$\begin{aligned}E_n^{(2)}(\lambda) = \frac{1}{4}\sum_{\mu\neq\pm\lambda}\frac{1}{\lambda^2-\mu^2}\frac{1}{N^2}\sum_{j,k} z_j z_k\Bigg\{&\left(\frac{\mathrm{i}\mu-\frac{\mathrm{i}}{2}}{\mathrm{i}\mu+\frac{\mathrm{i}}{2}}\right)^{j-k}\left[\left(\frac{\mathrm{i}\lambda+\frac{\mathrm{i}}{2}}{\mathrm{i}\lambda-\frac{\mathrm{i}}{2}}\right)^{j-k} + \left(\frac{\mathrm{i}\lambda-\frac{\mathrm{i}}{2}}{\mathrm{i}\lambda+\frac{\mathrm{i}}{2}}\right)^{j-k}\right] \\ &\pm\left(\frac{\mathrm{i}\mu+\frac{\mathrm{i}}{2}}{\mathrm{i}\mu-\frac{\mathrm{i}}{2}}\right)^{j-k}\left[K_-\left(\frac{\mathrm{i}\lambda+\frac{\mathrm{i}}{2}}{\mathrm{i}\lambda-\frac{\mathrm{i}}{2}}\right)^{j+k} + K_+\left(\frac{\mathrm{i}\lambda-\frac{\mathrm{i}}{2}}{\mathrm{i}\lambda+\frac{\mathrm{i}}{2}}\right)^{j+k}\right]\Bigg\}.\end{aligned}$$

$K_\pm$ is a quotient of Fourier transforms

$$K_\pm = \sqrt{\frac{\sum_{j=1}^{N} z_j \left(\frac{\mathrm{i}\lambda \pm \frac{\mathrm{i}}{2}}{\mathrm{i}\lambda \mp \frac{\mathrm{i}}{2}}\right)^{2j}}{\sum_{j=1}^{N} z_j \left(\frac{\mathrm{i}\lambda \mp \frac{\mathrm{i}}{2}}{\mathrm{i}\lambda \pm \frac{\mathrm{i}}{2}}\right)^{2j}}}.$$

The sum over μ can be transformed for $N \to \infty$ into a principle-value integral

$$E_n^{(2)}(\lambda) = \frac{1}{8\pi} \sum_{j,k} \frac{z_j z_k}{N} vp \int_{-\infty}^{+\infty} \frac{1}{\lambda^2 - \mu^2} \Bigg\{ \left(\mu + \frac{\mathrm{i}}{2}\right)^{(j-k-1)} \left(\mu - \frac{\mathrm{i}}{2}\right)^{(k-j-1)}$$
$$\times 2\cos[p(\lambda)(j-k)] \pm \left(\mu - \frac{\mathrm{i}}{2}\right)^{(j-k-1)} \left(\mu + \frac{\mathrm{i}}{2}\right)^{(k-j-1)}$$
$$\times [K_+ \mathrm{e}^{-\mathrm{i}p(\lambda)(j+k)} + K_- \mathrm{e}^{\mathrm{i}p(\lambda)(j+k)}] \Bigg\} \,\mathrm{d}\mu.$$

This principle-value integral can be evaluated by deforming the integration contour into the complex plane, closing it at infinity, which is possible as the integrand vanishes as r^{-4} at infinity. Thus only the pole structure of the integral matters.

There are poles at $\lambda = \pm\mu$ for all values of j, k, at $\mu = \frac{\mathrm{i}}{2}$ for $j > k$ in the first term and for $j < k$ in the second term and at $\mu = -\frac{\mathrm{i}}{2}$ with j, k dependence of the first and second term interchanged. It is convenient to split the sum over j, k into three parts:

$$\sum_{j,k} = \sum_{j=k} + \sum_{j>k} + \sum_{j<k}.$$

For each sum the contour can be deformed in such a way that the integrand only contains poles of first order, for which the residues are easily calculated.

The result of the integration is

$$E_n^{(2)}(\lambda) = -E^0(\lambda) \Bigg\{ \frac{1}{2} \sum_j \frac{z_j^2}{N} [2 \pm f(\lambda, j=k)] - \frac{2}{\lambda} \sum_{j>k} \frac{z_j z_k}{N} \sin[p(\lambda)(j-k)]$$
$$[2\cos[p(\lambda)(j-k)] \pm f(\lambda, j>k)] \Bigg\} \tag{53}$$

with $f(\lambda, j, k) = [K_+ \mathrm{e}^{-\mathrm{i}p(\lambda)(j+k)} + K_- \mathrm{e}^{\mathrm{i}p(\lambda)(j+k)}]$.

For the second-lowest excitation (two magnons) the computation is more involved, but still elementary, so we only give the result for the first-order correction to the energy of the two-magnon state:

$$E^{(1)}(\mu_1, \mu_2) = E^{(0)}(\mu_1, \mu_2) \Bigg\{ \sum_{j=1}^{N} \frac{z_j}{N} \pm 2 \Bigg[\sum_{j,k=1}^{N} \frac{z_j z_k}{N^2} (\exp[-2\mathrm{i}p_0(\mu_1)(j-k)]$$
$$+ \exp[-2\mathrm{i}p_0(\mu_2)(j-k)]) \pm 2E^{(0)}(\mu_1)E^{(0)}(\mu_2)$$
$$\times \Bigg[\sum_{j,k=1}^{N} \frac{z_j z_k}{N^2} \exp[-2\mathrm{i}p_0(\mu_1)(j-k)]$$
$$\times \sum_{j,k=1}^{N} \frac{z_j z_k}{N^2} \exp[-2\mathrm{i}p_0(\mu_2)(j-k)] \Bigg]^{\frac{1}{2}} \Bigg]^{\frac{1}{2}} \Bigg\}.$$

Furthermore there exist complex solutions of the Bethe ansatz equations. They describe bound states [18] with momentum

$$e^{ip(x)} = \left(\frac{x+i}{x-i}\right)$$

and energy

$$E^0_{\rm String}(x) = \frac{1}{x^2+1}$$

where x denotes the centre of the bound state.

The first-order correction for the two magnon bound state is

$$E^{(1)}_{\rm String}(x) = E^{(0)}_{\rm String}(x)\left[\left(\frac{1}{N}\sum_{j=1}^{N} z_j\right) \pm E^0_{CM}\sqrt{\sum_{j,k=1}^{N}\frac{z_j z_k}{N^2}\exp(-2ip(x)(j-k))}\right] \tag{54}$$

with $E^{(0)}_{CM} = \frac{1}{2}\frac{1}{x^2+\frac{1}{4}}$ the energy of the centre of the bound state.

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